COMPLEX GROUPS

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PREFACE

The notion of complex groups, believed to be new, lends itself to wide exploitation. Any complex group may be regarded as a generalized quotient group. Since quotient groups enjoy a prominent role in group theory, it is felt that complex groups may be worthy of attention.

Chapter One is devoted to the study of subsets, or complexes, of an arbitrary group. Some properties of these complexes are revealed, and a method for combining two complexes is defined. An algebra for this binary mode of combination is developed and used for defining complex groups.

In Chapter Two appear the most general theorems on complex groups. Theorem 2.22 provides a powerful method for constructing examples of various complex groups, and Theorem 2.20 is of fundamental importance for analyzing particular kinds of complex groups.

Chapter Three deals with three specific types of complex groups. A complete solution is obtained in Theorem 3.6 for elementary complex groups. Partial solutions are found for finite complex groups and linearly ordered complex groups. The best result for finite complex groups appears in Theorem 3.7.

A shift of emphasis is encountered in Chapter Four. Instead of placing restrictions upon a complex group, the restrictions are placed upon the group from which a complex group is formed. A complete characterization of all complex groups from an infinite cyclic group is given in Theorem 4.8 and Theorem 4.16.

One of the many possible questions which may arise in connection with complex groups is considered in Chapter Five. A divisible property is defined, and an attempt is made to discover which group properties are also divisible properties. Theorem 5.4 ensues as a by-product of this investigation, and throws some light on the nature of complex groups from any torsion group.

TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS ii

PREFACE iii

Chapter

One. COMPLEXES OF A GROUP 1

Two. GENERAL COMPLEX GROUPS 10

Three. SOME PARTICULAR COMPLEX GROUPS 22

Four. COMPLEX GROUPS FROM THE INFINITE CYCLIC GROUP 28

Five. DIVISIBLE PROPERTIES 39

BIBLIOGRAPHY 43

BIOGRAPHICAL SKETCH 44

CHAPTER ONE

COMPLEXES OF A GROUP

Definition 1.1. A complex of a group G is any subset of G. In particular, the null set $\emptyset$ is the null complex of group G.

Definition 1.2. If A and B are complexes of a group G, then the complex product AB of A and B is the set of all elements of the form ab, where $a \in A$ and $b \in B$. The formation of complex products is termed complex multiplication, and A and B are called the factors of the product AB.

In this paper the symbol J always denotes the additive group of rational integers.

Example 1.1. If $A = \{x \mid 2 \leq x \leq 4\}$ and $B = \{x \mid 3 \leq x \leq 5\}$ are complexes of group J, then $AB = \{x \mid 5 \leq x \leq 9\}$ is the complex product of A and B.

If a factor of a complex product is the null complex, then the product itself is the null complex.

Braces are suppressed when a factor appearing in a complex product is a single-element set. For example, if $\{a\}$ and B are complexes of a group G, then the complex product of $\{a\}$ and B is written aB.

Definition 1.3. A complex group G' from a group G is any set of complexes of G which themselves form a group under complex multiplication.

Example 1.2. Any quotient group of a group G is a complex group G' from group G.

The preceding definition and example are premature in the sense that they are not needed for the development of the present chapter. However, it is well to realize at this time that the elements of a complex group are complexes from some group. Since an understanding of any group is often gained through an understanding of its elements, a study of arbitrary complexes may properly precede a study of complex groups.

Throughout this chapter the symbols A, B, C, and D denote arbitrary complexes of any group.

The next theorem is an immediate consequence of the fact that every group is associative.

Theorem 1.1. $A(BC) = (AB)C$.

Complex multiplication is commutative when the complexes are restricted to some abelian group. It is possible to select complexes A and B from a non-abelian group such that $AB = BA$, but in general, complex multiplication is not commutative.

Complex multiplication is distributive over set-theoretic join both from the left and from the right. This is the statement of the next theorem and is easily proved.

Theorem 1.2. $A(B \cup C) = AB \cup AC$ and $(B \cup C)A = BA \cup CA$.

The notation $A \subseteq B$ is used to indicate that A is a subset of B. When A is a proper subset of B, the notation $A \subset B$ may be used.

Theorem 1.3. $A(B \cap C) \subseteq AB \cap AC$ and $(B \cap C)A \subseteq BA \cap CA$.

Proof. If $x \in A(B \cap C)$, then $x = ay$, where $a \in A$ and $y \in B \cap C$. Since $y \in B$ and $y \in C$, it follows that $ay \in AB$ and $ay \in AC$. Therefore, $A(B \cap C) \subseteq AB \cap AC$, and the first statement of the theorem is verified. A similar argument yields the second statement, and the theorem is proved.

The next example illustrates that complex multiplication is not distributive over set-theoretic meet.

The notation J_n is consistently used to denote the additive group of integers modulo n.

Example 1.3. If $A = \{1,5\}$, $B = \{0,3\}$, and $C = \{2,5\}$ are complexes of J_6, then $A(B \cap C) \subset AB \cap AC$.

The next theorem is used often to establish subsequent theorems. It is not difficult to prove.

Theorem 1.4. If $A \subseteq B$, then $CA \subseteq CB$ and $AC \subseteq BC$.

Theorem 1.5. If $A \subseteq B$ and $C \subseteq D$, then $AC \subseteq BD$.

Proof. If $A \subseteq B$, then $AC \subseteq BC$ by Theorem 1.4. Similarly, $C \subseteq D$ implies that $BC \subseteq BD$. Since $AC \subseteq BC$ and $BC \subseteq BD$, it follows that $AC \subseteq BD$, and the theorem is proved.

The notation $c(A)$ denotes the cardinal number of the complex A.

Theorem 1.6. If $A \neq \emptyset$ and $B \neq \emptyset$, then $c(AB) \geq c(A)$ and $c(AB) \geq c(B)$.

Proof. Let $b \in B$ and observe that if $Ab \subseteq AB$, then $c(Ab) \leq c(AB)$. Since $c(A) = c(Ab)$, it follows that $c(AB) \geq c(A)$, and the first part of the theorem is proved. The second part is proved in a similar manner.

The product $AA\ldots A$ is denoted by A^n, where n is a positive integer and A appears as a factor n times.

Definition 1.4. A non-null complex A of a group G is idempotent if $A^2 = A$.

Theorem 1.7. A finite complex A of a group G is idempotent if and only if A is a subgroup of G.

Proof. If A is a subgroup of group G, then A is obviously idempotent. Thus, let A be finite and idempotent, and observe that A = aA whenever $a \in A$. But if $a \in aA$, then A contains the identity element e of group G. Since $e \in A$, it follows that $e \in bA$ for each element b belonging to A, and therefore A contains the inverse of each of its elements. The above results combine to show that A is a subgroup, and the theorem is proved.

A modification of the previous theorem is usually encountered in the theory of finite groups. However, it is the complex, rather than the group, which must be finite to insure the validity of the theorem. This may be seen by referring to the next example.

The symbol R is used to denote the multiplicative group of positive real numbers.

Example 1.4. The infinite complex $A = \{x \mid x > 1\}$ is an idempotent complex of group R. Clearly A is not a subgroup of R because $1 \notin A$.

Definition 1.5. A non-null complex A of a group G is (properly) subpotent if A^2 is a (proper) subset of A.

Example 1.5. The complex $A = \{x \mid x \geq 2\}$ of group J is properly subpotent.

Definition 1.6. A non-null complex A of a group G is (properly) superpotent if A is a (proper) subset of A^2.

Example 1.6. The complex $A = \{x \mid x \geq -2\}$ of group J is properly superpotent.

Theorem 1.8. If A is a properly subpotent complex of a group G, then A is infinite.

Proof. Assume that A is finite. Then, since $A^2 \subset A$, it follows that $c(A^2) < c(A)$. But this is impossible because $c(A^2) \geq c(A)$ by Theorem 1.6, and the theorem is proved.

Unless otherwise stated, the symbol e denotes the identity element of the group G which is under immediate consideration.

Theorem 1.9. If A is a complex of a group G such that $e \in A$, then A is superpotent.

Proof. If $e \in A$, then $\{e\} \subseteq A$. By Theorem 1.4, $eA \subseteq A^2$, and A is superpotent as was to be proved.

Theorem 1.10. If A is a subpotent complex of a group G, then $A^n \subseteq A^m$, where m and n are positive integers such that $m < n$.

Proof. If A is subpotent, then $A^2 \subseteq A$. The relations $A^n \subseteq \ldots \subseteq A^m \subseteq \ldots \subseteq A^2 \subseteq A$ follow from repeated application of Theorem 1.4. It is seen that $A^n \subseteq A^m$, and the theorem is proved.

It should be noticed that every power A^k of a subpotent complex A is subpotent. This type of information proves to be useful when considering cyclic complex groups generated by a single complex.

The next theorem is analogous to Theorem 1.10 and is stated without proof.

Theorem 1.11. If A is a superpotent complex of a group G, then $A^m \subseteq A^n$, where m and n are positive integers such that $m < n$.

Every power A^k of a superpotent complex A is superpotent.

Definition 1.7. An element a of a group G has finite period p if p is the smallest positive integer such that $a^p = e$. If there is no such integer, then a has infinite period.

Theorem 1.12. If A is a subpotent complex of a group G such that $e \notin A$, then every element of A has infinite period.

Proof. Assume that there exists an element a belonging to A such that $a^p = e$ for some positive integer p. Then $e \in A^p$ because $a^p \in A^p$. But $A^p \subseteq A$ by Theorem 1.10, therefore $e \in A$. A contradiction has been found, and the theorem is proved.

Theorem 1.13. If a is an element of a properly subpotent complex A, then a has infinite period.

Proof. Observe that $e \notin A$ by Theorem 1.9. The theorem now follows by using Theorem 1.12.

Theorem 1.14. If the subpotent complexes A and B of a group G are commutative, then the product AB is subpotent.

Proof. If $A^2 \subseteq A$ and $B^2 \subseteq B$, then $A^2B^2 \subseteq AB$ by Theorem 1.5. Since $A^2B^2 = AABB = ABAB = (AB)^2$, it follows that $(AB)^2 \subseteq AB$, and the theorem is proved.

The next theorem is analogous to Theorem 1.14 and the proof is omitted.

Theorem 1.15. If the superpotent complexes A and B of a group G are commutative, then the product AB is superpotent.

The set of all subpotent complexes of an abelian group is closed under complex multiplication. A similar remark may be made for the set of all superpotent complexes of an abelian group.

Definition 1.8. Two complexes are comparable if one is a subset of the other.

Definition 1.9. Two complexes are intersecting if they are neither comparable nor disjoint.

Definition 1.10. A non-null complex A of a group G is dispotent if A and A^2 are disjoint.

Example 1.7. The set of all odd integers is a dispotent complex of group J.

Definition 1.11. A non-null complex A of a group G is interpotent if A and A^2 are intersecting.

Example 1.8. The complex $A = \{1,2\}$ is an interpotent complex of group J.

CHAPTER TWO

GENERAL COMPLEX GROUPS

Before proceeding it may be desirable to review the notion of a complex group given in Definition 1.3.

Definition 2.1. Any element of a complex group G' is said to be a member of G'.

Definition 2.2. A complex group is trivial if it contains precisely one member.

Example 2.1. The set whose sole element is the complex A defined in Example 1.4 is a trivial complex group from group R.

Definition 2.3. A complex group is discrete if each of its members is a single-element set.

Example 2.2. The set of all single-element complexes of any subgroup of a group G is a discrete complex group from group G.

From the foregoing it is seen that complex groups suffer from no lack of variety. It is not necessary to ask, for example, if there are such things as non-abelian complex groups. Their existence is guaranteed merely by the existence of non-abelian groups.

The notation G' is used to denote any complex group from some group G. The symbols X, Y, and Z denote members of G', and the symbol E denotes the identity member of G'. The symbols x, y, and z denote elements of G, and the symbol e represents the identity element of G.

The multiplicative notation of ordinary group theory is used for complex groups. Thus, X^{-1} is the unique member of G' such that $XX^{-1} = X^{-1}X = E$. As usual, $X^0 = E$, and $(X^n)^{-1} = X^{-n}$, where n is a positive integer.

In Chapter One the relationship between A and A^2 is used to classify an arbitrary complex A. In case A is a member of some complex group an alternate means of achieving the same classification is possible. This is the content of the next four theorems which, because of their simplicity, appear without proof.

Theorem 2.1. If X is a member of a complex group, then X is subpotent if and only if X is a subset of E.

Theorem 2.2. If X is a member of a complex group, then X is superpotent if and only if X is a superset of E.

Theorem 2.3. If X is a member of a complex group, then X is dispotent if and only if X and E are disjoint.

Theorem 2.4. If X is a member of a complex group, then X is interpotent if and only if X and E are intersecting.

It is clear by Theorem 2.1 and Theorem 2.2 that the identity member of a complex group is its only idempotent member.

Theorem 2.5. If X is a member of a complex group G' such that $X \neq E$ and X has finite period p, then X and E are not comparable.

Proof. If $X \subseteq E$, then X is subpotent by Theorem 2.1. Therefore $X^p \subseteq X$ by Theorem 1.10, and the relations $E = X^p \subseteq X \subseteq E$ follow. But this is impossible if $X \neq E$. Similarly, if $E \subseteq X$, then X is superpotent by Theorem 2.2. The relations $E \subseteq X \subseteq X^p = E$ then follow by Theorem 1.11. This is also impossible if $X \neq E$, and the theorem is proved.

Theorem 2.6. If X is a member of a complex group G' such that $e \in X$, then X is superpotent.

Proof. The theorem follows immediately, since $\{e\} \subseteq X$ implies that $\{e\}E \subseteq XE$.

Theorem 2.7. If X is a member of a complex group G' such that $X \neq E$ and $e \in X$, then X has infinite period.

Proof. If $e \in X$, then X is superpotent by Theorem 2.6. Since X and E are comparable, X cannot have finite period by Theorem 2.5. Therefore X must have infinite period, and the theorem is proved.

Definition 2.4. A complex group is linearly ordered if its members are pairwise comparable.

Example 2.3. Define an n-ray to be the set of all integers greater than or equal to integer n. The set J' of all n-rays for all integers n is a complex group from group J. Notice that J' is linearly ordered. The identity member of J' is $E = \{x | x \in J \text{ and } x \geq 0\}$, and the inverse of the member $X = \{x | x \in J \text{ and } x \geq 3\}$ is the member $X^{-1} = \{x | x \in J \text{ and } x \geq -3\}$.

Theorem 2.8. If X and Y are members of a complex group G', then $X \wedge Y \neq \emptyset$ if and only if $X^{-1} \wedge Y^{-1} \neq \emptyset$.

Proof. Assume that $X \wedge Y \neq \emptyset$. Since $X^{-1} \neq \emptyset$, it follows that $X^{-1}(X \wedge Y) \neq \emptyset$. But $X^{-1}(X \wedge Y) \subseteq E \wedge X^{-1}Y$, therefore $E \wedge X^{-1}Y \neq \emptyset$. Since $Y^{-1} \neq \emptyset$, it follows that $(E \wedge X^{-1}Y)Y^{-1} \neq \emptyset$. Thus, $Y^{-1} \wedge X^{-1} \neq \emptyset$ because $(E \wedge X^{-1}Y)Y^{-1} \subseteq Y^{-1} \wedge X^{-1}$. Due to the symmetric nature of the theorem, the proof is now complete.

The next theorem follows at once from the previous theorem.

Theorem 2.9. If X is a member of a complex group G', then $X \wedge E \neq \emptyset$ if and only if $X^{-1} \wedge E \neq \emptyset$.

It may be concluded from the preceding theorem that a dispotent member of a complex group always has a dispotent inverse.

Theorem 2.10. If X is a member of a complex group G' such that $X \wedge E \neq \emptyset$, then $X^n \wedge E \neq \emptyset$ for every integer n.

Proof. The theorem is true when $n = 0$. Therefore let n be a positive integer, and assume that $X \wedge E = M \neq \emptyset$. Then $M^n \subseteq X^n$, because $M \subseteq X$. Similarly, $M \subseteq E$ implies that $M^n \subseteq E^n = E$. Thus, $M^n \subseteq X^n \wedge E$, and since $M^n \neq \emptyset$, it follows that $X^n \wedge E \neq \emptyset$. In case n is a negative integer, Theorem 2.9 guarantees that $X^{-n} \wedge E \neq \emptyset$ if and only if $X^n \wedge E \neq \emptyset$. Since $-n$ is positive the above argument may be repeated, and the proof is complete.

The consideration of various cases encountered in the foregoing proof is unavoidable because the notation M^{-n} is meaningless when n is a positive integer and M is not a member of the complex group.

Theorem 2.11. If X and Y are members of a complex group G', then $X \subset Y$ if and only if $Y^{-1} \subset X^{-1}$.

Proof. If $X \subset Y$, then $E \subseteq X^{-1}Y$. It follows that $Y^{-1} \subseteq X^{-1}$, and since $X \neq Y$ it is known that $Y^{-1} \neq X^{-1}$. Therefore $Y^{-1} \subset X^{-1}$, and the theorem is proved.

The next theorem is an important special case of the theorem just proved.

Theorem 2.12. If X is a member of a complex group G', then $X \subset E$ if and only if $E \subset X^{-1}$.

The foregoing theorem states that a member X of a complex group is subpotent if and only if its inverse X^{-1} is superpotent. With the aid of Theorem 2.9 and Theorem 2.12 it may be seen that the inverse of an interpotent member of a complex group is always interpotent.

Theorem 2.13. If Y and Z are distinct members of a complex group G' such that $Y \cap Z \neq \emptyset$, then there exists a member X of G' such that $X \neq E$ and $X \cap E \neq \emptyset$.

Proof. If $Y \cap Z \neq \emptyset$, then $(Y \cap Z)Z^{-1} \neq \emptyset$. Thus $YZ^{-1} \cap E \neq \emptyset$, because $(Y \cap Z)Z^{-1} \subseteq YZ^{-1} \cap E$. The theorem follows by identifying X with YZ^{-1}.

Definition 2.5. A complex group is disjunctive if its members are pairwise disjoint.

Although any quotient group of a group G is a disjunctive complex group from G, the converse is not necessarily true. This is easily seen by returning to Example 2.1.

Theorem 2.14. If the identity member E of a complex group G' is a subgroup of group G, then G' is a disjunctive complex group.

Proof. Assume that G' is not disjunctive. Then there exist members Y and Z of G' such that $Y \cap Z \neq \emptyset$. By Theorem 2.13 there exists a member X of G' such that $X \cap E \neq \emptyset$. This implies that $X^{-1} \cap E \neq \emptyset$ by Theorem 2.9. Theorem 2.12 reveals that it is not the case that both $X \subseteq E$ and $X^{-1} \subseteq E$. Without loss of generality, suppose that X is not a subset of E. Then there exist elements x and y of G such that $x \in X - E$ and $y \in X^{-1} \cap E$. Now observe that $xy \notin E$, for if $xy \in E$, then $(xy)y^{-1} = x \in E$. But this is impossible because $x \in X - E$, therefore $xy \notin E$. On the other hand, since $x \in X$ and $y \in X^{-1}$, it follows that $xy \in XX^{-1} = E$. The statements $xy \notin E$ and $xy \in E$ are incompatible, and the theorem is proved.

Example 2.1 demonstrates that the converse of the preceding theorem is false.

Definition 2.6. A member of a complex group is free if its set-theoretic meet with each of the remaining members is null.

Every member of a disjunctive complex group is free. No member of a linearly ordered complex group is free.

Theorem 2.15. A complex group is disjunctive if and only if it possesses a free member.

Proof. Assume that Y is a free member of a complex group G' which is not disjunctive. By Theorem 2.13 there exists a member X of G' such that $X \cap E \neq \emptyset$. Since $Y \neq \emptyset$, it follows that $Y(X \cap E) \neq \emptyset$. But $Y(X \cap E) \subseteq YX \cap Y$, therefore $YX \cap Y \neq \emptyset$. This contradicts the hypothesis that Y is free, and the theorem is proved.

Theorem 2.16. If X is a member of a complex group G' such that $X \neq E$, then $e \in X$ if and only if $e \notin X^{-1}$.

Proof. If $e \in X$ and $e \in X^{-1}$, then both X and X^{-1} are superpotent by Theorem 2.6. From Theorem 2.12 it is known that X is subpotent if and only if X^{-1} is superpotent. Since $X \neq E$, X cannot be both subpotent and superpotent, therefore the theorem is proved.

Theorem 2.17. If X is a member of a complex group G' such that $X \cap E \neq \emptyset$, then $X \cap E$ is subpotent.

Proof. Assume that $X \cap E = M$ is non-null. Since $M \subseteq X$ and $M \subseteq E$, it follows that $M^2 \subseteq EX = X$ by Theorem 1.5. On the other hand, $M \subseteq E$ implies that $M^2 \subseteq E^2 = E$. Thus, $M^2 \subseteq X \cap E = M$, and $X \cap E$ is subpotent as was to be proved.

Theorem 2.18. If X is a member of a complex group G' such that $X \neq E$ and $X \cap E \neq \emptyset$, then $X \cap E$ is infinite.

Proof. Suppose that $X \cap E$ is finite and non-null. By Theorem 2.17, $X \cap E$ is subpotent. Consequently, $X \cap E$ is a subgroup of group G by Theorem 1.7. Thus $e \in X \cap E$, and $X \cap E$ is superpotent by Theorem 1.9. However, $X \cap E$

is not both subpotent and superpotent because $X \neq E$, and the theorem is proved.

Theorem 2.19. If Y and Z are distinct members of a complex group G' such that $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is infinite.

Proof. If $Y \cap Z \neq \emptyset$, then $YZ^{-1} \cap E \neq \emptyset$ by Theorem 2.13. Theorem 2.18 reveals that $YZ^{-1} \cap E$ is infinite. Since $(YZ^{-1} \cap E)Z \subseteq Y \cap Z$, it follows that $Y \cap Z$ is infinite, and the theorem is proved.

Theorem 2.20. If X and Y are members of a complex group G', then $c(X) = c(Y)$.

Proof. By Theorem 1.6, $c(XX^{-1}) \geq c(X)$, therefore $c(E) \geq c(X)$. Similarly, $c(X) \geq c(E)$ because $c(XE) \geq c(E)$, and it is seen that $c(X) = c(E)$. An identical argument shows that $c(Y) = c(E)$, and the theorem is proved.

Theorem 2.21. If a complex group G' is not disjunctive, then all of its members are infinite.

Proof. Let X and Y be members of G' such that $X \cap Y \neq \emptyset$. Then $X \cap Y$ is infinite by Theorem 2.19. Thus X is infinite because $X \cap Y \subseteq X$. By Theorem 2.20 all members are infinite, and the theorem is proved.

Up to this point the complex groups presented in the various examples have been either disjunctive or linearly ordered. In order to construct complex groups which lie in neither of these two categories, it is first necessary to establish new definitions which lead to a helpful theorem.

Definition 2.7. The external direct product of the groups $G_1, \ldots, G_n$ is the set of all n-tuples of the form $(g_1, \ldots, g_n)$, where $g_i \in G_i$ for $i = 1, \ldots, n$, and where multiplication is defined by multiplying components. Notice that an external direct product is itself a group.

The notation $G_1 \times \cdots \times G_n$ is used to denote the external direct product of the groups $G_1, \ldots, G_n$.

Since complex groups are themselves groups, it is possible to form external direct products with complex groups. However, it is important to observe that if G_i' is a complex group from group G_i for $i = 1, \ldots, n$, then it is not the case that $G_1' \times \cdots \times G_n'$ is a complex group from $G_1 \times \cdots \times G_n$. This is easily seen by examining typical elements of the two external direct products in question.

Definition 2.8. The Cartesian image of an n-tuple $(S_1, \ldots, S_n)$ whose components are sets is the usual Cartesian product of the sets $S_1, \ldots, S_n$.

Definition 2.9. The reduced direct product of the complex groups $G_1', \ldots, G_n'$ is the set of Cartesian images of all the n-tuples which belong to the external direct product $G_1' \times \cdots \times G_n'$, where G_i' is a complex group from group G_i for $i = 1, \ldots, n$.

The notation $G_1' \otimes \cdots \otimes G_n'$ denotes the reduced direct product of the complex groups $G_1', \ldots, G_n'$.

The next theorem is straightforward and is stated without proof.

Theorem 2.22. If G_i' is a complex group from group G_i for $i = 1, \ldots, n$, then $G_1' \otimes \cdots \otimes G_n'$ is a complex group from group $G_1 \times \cdots \times G_n$.

It is now possible to construct a reduced direct product which fails to manifest all of the properties of its component complex groups. This is illustrated by the next example.

Example 2.4. Let J_3' be the unique 3-element complex group from group J_3. Let J' be the complex group from group J described in Example 2.3. By the previous theorem, $J' \otimes J_3'$ is a complex group from group $J \times J_3$. Although J' is linearly ordered and J_3' is disjunctive, $J' \otimes J_3'$ is neither linearly ordered nor disjunctive. This may be seen by considering the three members

$E = \{(x,y) | x \in J \text{ and } x \geq 0 \text{ and } y \in J_3 \text{ and } y = 0\}$,

$X = \{(x,y) | x \in J \text{ and } x \geq 1 \text{ and } y \in J_3 \text{ and } y = 0\}$, and

$Y = \{(x,y) \mid x \in J$ and $x \geq 1$ and $y \in J_3$ and $y = 1\}$. Since $X \subset E$, $J' \otimes J_3'$ is not disjunctive, and since $X \cap Y = \emptyset$, $J' \otimes J_3'$ is not linearly ordered. Notice that E is the identity member of $J' \otimes J_3'$.

In the next example a reduced direct product is used to demonstrate that the converse of Theorem 2.6 is false.

Example 2.5. Let J and J' be the group and complex group described in the previous example. Let R and R' be the group and complex group defined in Example 2.1. Then $J' \otimes R'$ is a complex group from group $J \times R$. The identity element of $J \times R$ is $e = (0,1)$, and the identity member of $J' \otimes R'$ is $E = \{(x,y) \mid x \in J$ and $x \geq 0$ and $y \in R$ and $y > 1\}$. Notice that e is not an element of E. Next, consider the member $X = \{(x,y) \mid x \in J$ and $x \geq -1$ and $y \in R$ and $y > 1\}$. Since $E \subset X$, it follows that X is superpotent, and thus the converse of Theorem 2.6 is seen to be false.

CHAPTER THREE

SOME PARTICULAR COMPLEX GROUPS

Definition 3.1. A complex group G' is elementary if each of its members is finite.

A complex group from a finite group is always elementary. Notice that an elementary complex group may possess infinitely many members. This may be seen in the following example.

Example 3.1. If G is an infinite group possessing a finite normal subgroup N, then the quotient group G/N is an elementary complex group which has infinitely many members.

The next six theorems effect a complete characterization of the elementary complex groups.

Theorem 3.1. If G' is an elementary complex group from group G, then E is a subgroup of G.

Proof. Since E is finite and idempotent, the theorem follows at once from Theorem 1.7.

Theorem 3.2. If G' is an elementary complex group from group G such that $x \in X$ and $y \in Y$, then $xY = Xy = XY$.

Proof. Observe that $c(Y) = c(xY)$. Also notice that $c(Y) = c(XY)$ by Theorem 2.20, therefore $c(xY) = c(XY)$. Since XY is finite and $xY \subseteq XY$, it follows that $xY = XY$. The remaining equality is obtained in a similar manner, and the theorem is proved.

Theorem 3.3. If G' is an elementary complex group from group G, then the set-theoretic join U of all its members is a subgroup of G.

Proof. Since E is a subgroup of G by Theorem 3.1, and since $E \subseteq U$, it follows that $e \in U$. To show that U is closed, assume that $x,y \in U$, and notice that there exist members X and Y of G' such that $x \in X$ and $y \in Y$. Since $xy \in XY \subseteq U$, $xy \in U$ and U is closed. To show that each element x of U has an inverse x^{-1} in U, observe that there exists a member X of G' such that $x \in X$. By Theorem 3.2, $XX^{-1} = xX^{-1}$, hence there exists an element x^{-1} of X^{-1} such that $xx^{-1} = e$. Since $X^{-1} \subseteq U$, it follows that $x^{-1} \in U$. Thus U satisfies the group postulates, and the theorem is proved.

It may be seen by returning to Example 2.1 that neither Theorem 3.1 nor Theorem 3.3 is true for an arbitrary complex group G'. The next example shows that Theorem 3.2 also fails when it is not stipulated that G' is elementary.

Example 3.2. Let J' be the linearly ordered complex group described in Example 2.3. If $X = \{z | z \geq 2\}$, $Y = \{z | z \geq 3\}$, and $x = 4$, then $x \in X$ but $xY \neq XY$.

Theorem 3.4. If G' is an elementary complex group from group G, then E is a normal subgroup of the set-theoretic join U of all the members of G'.

Proof. If $x \in U$, then there exists a member X of G' such that $x \in X$. By Theorem 3.2, $XE = xE$ and $EX = Ex$. Since $XE = EX$, $xE = Ex$, and E is a normal subgroup of group U as was to be proved.

Theorem 3.5. If X is a member of an elementary complex group G', then X is both a left and right coset of the identity member E.

Proof. If $x \in X$ and $X \in G'$, then $X = xE = Ex$ because $X = XE = EX$, and the theorem is proved.

Every elementary complex group is disjunctive. This result was actually obtained earlier, and is the contrapositive of Theorem 2.21.

The next theorem is merely a summary of the foregoing results on elementary complex groups, and is presented without further proof.

Theorem 3.6. An elementary complex group G' from a group G is a quotient group U/E, where U is the set-theoretic join of the members of G' and E is the identity member of G'.

Complex groups may be regarded as generalized quotient groups. Indeed, any quotient group is a complex group, but complex groups are not always quotient groups. Under suitable restrictions, however, the more general notion of complex groups reduces to the familiar notion of quotient groups. This, of course, is the case when the complex groups are elementary.

Definition 3.2. A complex group is finite if it has finitely many members. The members themselves may be infinite.

Although the preceding classification is natural, it does not help to produce many theorems. There are only a few statements true of finite complex groups which are not also true of arbitrary complex groups. The requirement of finiteness does not seem to simplify the theory. The reason for this is partly indicated by the next example.

Example 3.3. Construct in the group $W = J \times R$ the three complexes

$$A = \{(x,1) \mid x \in J \text{ and } x \text{ is even and } 1 \in R\},$$

$$B = \{(x,1) \mid x \in J \text{ and } x \text{ is odd and } 1 \in R\}, \text{ and}$$

$$C = \{(x,y) \mid x \in J \text{ and } y \in R \text{ and } y > 1\}.$$

If $E = A \cup C$ and $X = B \cup C$, then the 2-element set whose members are E and X is a complex group W' from group W. Notice that the two members of W' are intersecting.

The next theorem represents the best result for finite complex groups.

Theorem 3.7. No two members of a finite complex group G' are comparable.

Proof. If X and Y are distinct members of G', then $XY^{-1} \neq E$. Since G' is finite, XY^{-1} has finite period. By Theorem 2.5 it is not the case that $XY^{-1} \subset E$ or $E \subset XY^{-1}$. Therefore, it is not the case that $X \subset Y$ or $Y \subset X$, and the theorem is proved.

Definition 3.3. A group is a *torsion group* if each of its elements has finite period.

Definition 3.4. A group is *torsion free* if each of its elements different from the identity element has infinite period.

From Theorem 3.7 it may be observed that every non-trivial linearly ordered complex group is infinite. Since a linearly ordered complex group possesses no non-trivial finite subgroup, it follows that every linearly ordered complex group is torsion free. It is not the case, however, that every linearly ordered complex group is an infinite cyclic group. This is illustrated in the next example.

Example 3.4. Let Q be the additive group of all real numbers. Define an r-ray to be the set of all real numbers greater than r, where r is a real number. The set Q' of all r-rays for all real numbers r is a complex group from group Q. Although Q' is a linearly ordered complex group, it is not an infinite cyclic group.

CHAPTER FOUR

COMPLEX GROUPS FROM THE INFINITE CYCLIC GROUP

An interesting problem is that of determining what kinds of complex groups are possessed by a given group. When the given group is an infinite cyclic group, the problem is readily solved. Thus, it is partly known what kinds of complex groups are possessed by any non-torsion group, since a non-torsion group always has an element which generates an infinite cyclic subgroup.

Any infinite cyclic group is isomorphic to the additive group J of integers. From a notational viewpoint, it is less cumbersome to prove theorems about group J whose elements are integers, than to prove theorems about an abstract infinite cyclic group whose elements are powers of some generating element. For this reason in the present chapter a departure is made from the multiplicative notation. The additive notation for groups is used throughout the chapter, and theorems which are valid for any infinite cyclic group appear merely as theorems about the particular group J. Clearly, no loss of generality is involved in this procedure.

Theorem 4.1. If A is an idempotent complex of group J which contains at least one positive integer, then A contains infinitely many positive integers.

Proof. Assume that A contains finitely many positive integers and let p denote the maximum such integer. Since $p > 0$, it follows that $2p > p$. But $2p \leq p$, because $2p = p + p \in A + A = A$. Thus the assumption that A possesses a maximum positive integer is false, and the theorem is proved.

The next theorem is exactly analogous to the preceding theorem, and the proof is omitted.

Theorem 4.2. If A is an idempotent complex of group J which contains at least one negative integer, then A contains infinitely many negative integers.

Theorem 4.3. If J' is an elementary complex group from group J, then J' is discrete.

Proof. The identity member E of J' is finite by definition. Furthermore, E contains no negative or positive integers by Theorem 4.1 and Theorem 4.2. Thus $E = \{0\}$, and by Theorem 2.20, J' is discrete as was to be proved.

Definition 4.1. A full ray is a complex of group J which contains infinitely many positive and infinitely many negative integers.

Theorem 4.4. If A is an idempotent full ray of group J, then A is an infinite cyclic subgroup of J.

Proof. Let p denote the smallest positive integer of A and q the largest negative integer of A. Then it is not the case that $p < -q$, for this implies that $q < p + q < 0$, contradicting the fact that q is the largest negative integer of A. Nor is it the case that $-q < p$, for this implies that $0 < p + q < p$, contradicting the fact that p is the smallest positive integer of A. Therefore, $q = -p$, and since A is idempotent, it follows that $np \in A$ for every integer n. It remains to show that A contains only the integral multiples of p. Suppose that there exists an integer r belonging to A such that $np < r < (n + 1)p$ for some integer n. The result of adding $-np$ to each term reveals that $0 < r - np < p$, once again contradicting the fact that p is the smallest positive integer of A. Thus A is an infinite cyclic subgroup generated by p, and the theorem is proved.

Theorem 4.5. If J' is a finite and non-trivial complex group from group J, then E is an infinite cyclic subgroup of J.

Proof. Observe that E is not finite, for in this case $E = \{0\}$ by Theorem 4.3, and J' is either infinite or trivial. If E has both positive and negative integers, then the theorem follows from Theorem 4.4. Hence, assume that E contains no negative integers, and let X be a member of J' distinct from E with period k. Then X contains no negative integers, for suppose that $q < 0$ and $q \in X$. Since $kX = E$, it follows that $kq \in E$. But this is impossible because $kq < 0$. It is also evident that $0 \notin X$, for if $0 \in X$, then $0 + E \subseteq X + E$, and this is impossible by Theorem 3.7. Finally, if X contains only positive integers, then there is a minimum such integer p. Since $kX = E$, it follows that kp is the minimum integer of E. This is also impossible, since $kp > 0$ and E is idempotent. Therefore it is seen that E is a full ray of group J, and the theorem is proved.

It is a matter of routine to verify that if H is an infinite subgroup of group J, then the distinct cosets of H are of the form $x + H$, where $0 \leq x < p$ and p is the smallest positive integer of H. The next theorem is a weaker version of this statement, and appears without proof.

Theorem 4.6. If E is an infinite subgroup of group J, then E possesses a finite number of cosets.

It is now clear that if J' is a finite and non-trivial complex group from group J, then E is an infinite cyclic subgroup of J which possesses a finite number of pairwise disjoint cosets. However, it may not logically be concluded at this point that J' is a quotient group of some subgroup of J. Conceivably, a member X of J' could be the join of two or more cosets of E. The next theorem demonstrates that this is not the case.

Theorem 4.7. If J' is a finite and non-trivial complex group from group J, then each member X of J' is a coset of E.

Proof. Let $X = \{x_1, x_2, x_3, \ldots\}$ be any member of J'. It is clear that X is the join of one or more cosets of E, since $X = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots$, and $X = X + E = (\{x_1\} + E) \cup (\{x_2\} + E) \cup (\{x_3\} + E) \cup \ldots$. Now suppose that E_1 and E_2 are two distinct cosets of E such that $E_1 \subseteq X$ and $E_2 \subseteq X$. Let E_3 be a coset of E such $E_3 \subseteq -X$. It follows that $E_1 + E_3 \subseteq X + (-X) = E$ and $E_2 + E_3 \subseteq X + (-X) = E$. But $E_1 + E_3$ and $E_2 + E_3$ are distinct cosets of E, and it is impossible for E to contain two distinct cosets of itself. Therefore each member of J' is a coset of E, and the theorem is proved.

The next theorem is a recapitulation of the foregoing results.

Theorem 4.8. If J' is a finite and non-trivial complex group from group J, then J' is a finite quotient group of some subgroup of J.

Definition 4.2. A right ray is an infinite complex of group J which contains at most finitely many negative integers. A left ray is an infinite complex of group J which contains at most finitely many positive integers. A ray is a complex of group J which is either a left ray or a right ray.

Theorem 4.9. If J' is an infinite and non-discrete complex group from group J, then the identity member E of J' is a ray.

Proof. Since J' is not discrete, E is infinite. If E is not a ray, then E must be a full ray. By Theorem 4.4, it then follows that E is an infinite cyclic subgroup of J. By Theorem 4.6, E possesses a finite number of cosets, therefore J' is finite. This contradicts the hypothesis that J' is infinite, and the theorem is proved.

When it is necessary to assume that E is a ray in the following theorems and examples, it is assumed that E is a right ray without loss of generality. Thus, all ensuing statements involving the notion of right rays give rise to analogous statements involving the notion of left rays.

Theorem 4.10. If J' is an infinite complex group from group J such that E is a right ray, then the minimum integer belonging to E is 0.

Proof. E does not contain negative integers, for this implies that E is a full ray by Theorem 4.2. Since E is idempotent, the minimum integer of E cannot be a positive integer. Therefore, the minimum integer of E is 0, as was to be proved.

Theorem 4.11. If J' is an infinite complex group from group J such that E is a right ray, then each member of J' is a right ray.

Proof. Let X be a member of J' distinct from E. If $x \in X$, then $x + E$ has infinitely many positive integers. But $x + E \subseteq X + E = X$, therefore X contains infinitely many positive integers. Now assume that X also contains infinitely many negative integers. Suppose that $y \in -X$, and notice that $y + X$ has infinitely many negative integers. But $y + X \subseteq (-X) + X = E$, therefore E has infinitely many negative integers. This contradicts the hypothesis that E is a right ray, and the theorem is proved.

The next two theorems are easily established, and their proofs are withheld.

Theorem 4.12. If J' is an infinite complex group from group J such that E is a right ray, and if X and Y are members of J' whose respective minimum integers are x and y, then the minimum integer of X + Y is x + y.

Theorem 4.13. If J' is an infinite complex group from group J such that E is a right ray, then the minimum integer of a member X of J' is x if and only if the minimum integer of -X is -x.

The conjecture that every infinite and non-discrete complex group from group J is linearly ordered is false. This statement is supported by the next example.

Example 4.1. Let J' denote the set of all complexes in group J of the form $\{n\} \cup \{x | x \geq n + 3\}$, where n is permitted to take on all integral values. Then J' is an infinite complex group from group J which is not linearly ordered. The identity member of J', obtained by setting n equal to 0, is $E = \{0\} \cup \{x | x \geq 3\}$. Notice that E is an idempotent right ray whose minimum integer is 0, in accordance with the related theorems. Four typical members of J', obtained by setting n equal to 1, 2, -1, and 3 respectively, are

$$X = \{1\} \cup \{x | x \geq 4\},$$

$$2X = \{2\} \cup \{x | x \geq 5\},$$

$$-X = \{-1\} \cup \{x | x \geq 2\}, \text{ and}$$

$$3X = \{3\} \cup \{x \mid x \geq 6\}.$$

It is easily seen that X and 2X are intersecting members of J', therefore J' is not linearly ordered. The members -X and 3X are listed to help illustrate Theorem 4.12 and Theorem 4.13.

Theorem 4.14. If J' is an infinite complex group from group J such that E is a right ray, and if x is the minimum integer of member X of J', then $X = x + E$.

Proof. Since $x + E \subseteq X + E = X$, it follows that $x + E \subseteq X$. By Theorem 4.13, $-x \in -X$, and since $-x + X \subseteq -X + X = E$, it follows that $-x + X \subseteq E$. But if $-x + X \subseteq E$, then $X \subseteq x + E$. The relations $x + E \subseteq X$ and $X \subseteq x + E$ combine to give the theorem.

Theorem 4.15. If J' is an infinite complex group from group J such that E is a right ray, then the set H of minimum integers of all members of J' is an infinite cyclic subgroup of J.

Proof. By Theorem 4.10 the minimum integer of E is 0, therefore $0 \in H$. If $x, y \in H$, then there exist members X and Y of J' with minimum integers x and y respectively. By Theorem 4.12 the minimum integer of $X + Y$ is $x + y$, therefore $x + y \in H$, and H is closed under addition. If $z \in H$, then there exists a member Z of J' whose minimum integer is z. But the minimum integer of -Z is -z by Theorem 4.13, and therefore H contains the inverse of each of its elements. Associativity of H is guaranteed by the associativity of J, and the theorem is proved.

For convenience, the foregoing results are collected in the next theorem.

Theorem 4.16. If J' is an infinite and non-discrete complex group from group J, and if the identity member E of J' is a right ray, then there exists an infinite cyclic subgroup H of J such that the members of J' are precisely the complexes of the form h + E, where h is an element of H.

The next example indicates how Theorem 4.16 may be used to construct an infinite and non-discrete complex group from group J.

Example 4.2. The identity member E of the complex group to be constructed may be any idempotent right ray. As an example, suppose that $E = \{0,3\} \cup \{x | x \geq 5\}$. Notice that E has the desired properties. Next it is necessary to select any infinite subgroup H of group J. Suppose that $H = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$. The members of the complex group then have the form h + E, where h ranges over H. A few representative members are

$$0 + E = \{0,3\} \cup \{x | x \geq 5\},$$

$$3 + E = \{3,6\} \cup \{x \mid x \geq 8\},$$
$$-3 + E = \{-3,0\} \cup \{x \mid x \geq 2\},$$
$$6 + E = \{6,9\} \cup \{x \mid x \geq 11\}, \text{ and}$$
$$-6 + E = \{-6,-3\} \cup \{x \mid x \geq -1\}.$$

CHAPTER FIVE

DIVISIBLE PROPERTIES

Definition 5.1. A property P of a group G is divisible if every complex group G' from group G possesses property P.

The next theorem is obvious, and appears without proof.

Theorem 5.1. Any complex group from an abelian group is abelian.

Thus, the property of being abelian is a divisible property. The main objective of the present chapter is to tabulate some of the less evident divisible properties of groups.

Theorem 5.2. Any complex group G' from a cyclic group G is cyclic.

Proof. It is well known that any quotient group of a cyclic group is cyclic. Therefore, it is only necessary to consider the infinite and non-discrete complex groups from an infinite cyclic group. The results of such considerations are already embodied in Theorem 4.16, where it is obvious that G' is isomorphic to the subgroup H. Since H is itself cyclic, the theorem is proved.

Theorem 5.3. Any complex group G' from a torsion group G is a torsion group.

Proof. Suppose that G is a torsion group admitting of a complex group G' which is not a torsion group. Then G' possesses a member X which has infinite period. Assume that p is the finite period of x, where x is any element of X. Since $x^p = e \in X^p$, it follows by Theorem 2.6 that $E \subseteq X^p$. By Theorem 2.12, $X^{-p} \subseteq E \subseteq X^p$. If $y \in X^{-p}$, and if q is the finite period of y, then $y^q = e \in (X^{-p})^q$. But the relations $E \supseteq X^{-p} \supseteq (X^{-p})^2 \supseteq \ldots \supseteq (X^{-p})^q$ ensue from repeated use of Theorem 1.4, therefore $e \in X^{-p}$. Theorem 2.16 asserts that it is not the case that both $e \in X^{-p}$ and $e \in X^p$. The assumption that G' is not a torsion group is therefore false, and the theorem is proved.

It is now easy to further characterize the complex groups from any torsion group.

Theorem 5.4. Any complex group G' from a torsion group G is disjunctive.

Proof. Assume that G' is a non-disjunctive complex group from torsion group G. Then there exist distinct members Y and Z of G' such that $Y \cap Z \neq \emptyset$. By Theorem 2.13

there exists a member X belonging to G' such that $X \cap E \neq \emptyset$. Since G is a torsion group, it follows by Theorem 5.3 that G' is also a torsion group. Therefore X has finite period, and by Theorem 2.7, $e \notin X$. By Theorem 2.17, $X \cap E = M$ is subpotent. Since M is subpotent and $e \notin M$, every element of M has infinite period by Theorem 1.12. This contradicts the hypothesis that G is a torsion group, and the theorem is proved.

Definition 5.2. A group is of bounded order b if each of its elements has period less than or equal to b, and at least one of its elements has period b.

Theorem 5.5. If G is a group of bounded order b, then any complex group G' from group G is of bounded order less than or equal to b.

Proof. Suppose that group G of bounded order b has a complex group G' which is not of bounded order less than or equal to b. Since G is a torsion group, it follows by Theorem 5.3 that G' is a torsion group. Let p be the period of a member X of G', and assume that $p > b$. Let q be the period of any element x of X, and notice that $x^q = e$ and $q \leq b$. Since $x^q = e \in X^q$ and $X^q \neq E$, it follows from Theorem 2.6 that $E \subset X^q$. However, Theorem 5.4 asserts that G' is disjunctive, therefore the assumption that $p > b$ is false, and the theorem is proved.

Definition 5.3. A torsion group G is a p-primary group if there exists a prime number p such that each element of G has period a power of p.

The proof for the final theorem is withheld because it is merely a slight variation of the proof for the preceding theorem.

Theorem 5.6. Any complex group G' from a p-primary group G is p-primary.

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BIOGRAPHICAL SKETCH

Wade Lee Allen was born February 1, 1928, in Cleveland, Ohio. His undergraduate work was pursued at Jacksonville Junior College, George Washington University, and Ohio University. In 1953 he graduated with honors in mathematics from Ohio University, receiving a Bachelor of Science degree.

He then came to the University of Florida where he was both a teaching assistant and graduate student for the next two years. During the following year he served as an interim instructor, then he accepted a job with California Institute of Technology as a research engineer at their Jet Propulsion Laboratory. After serving in this capacity for one year, he returned to the University of Florida to resume his studies in mathematics.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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